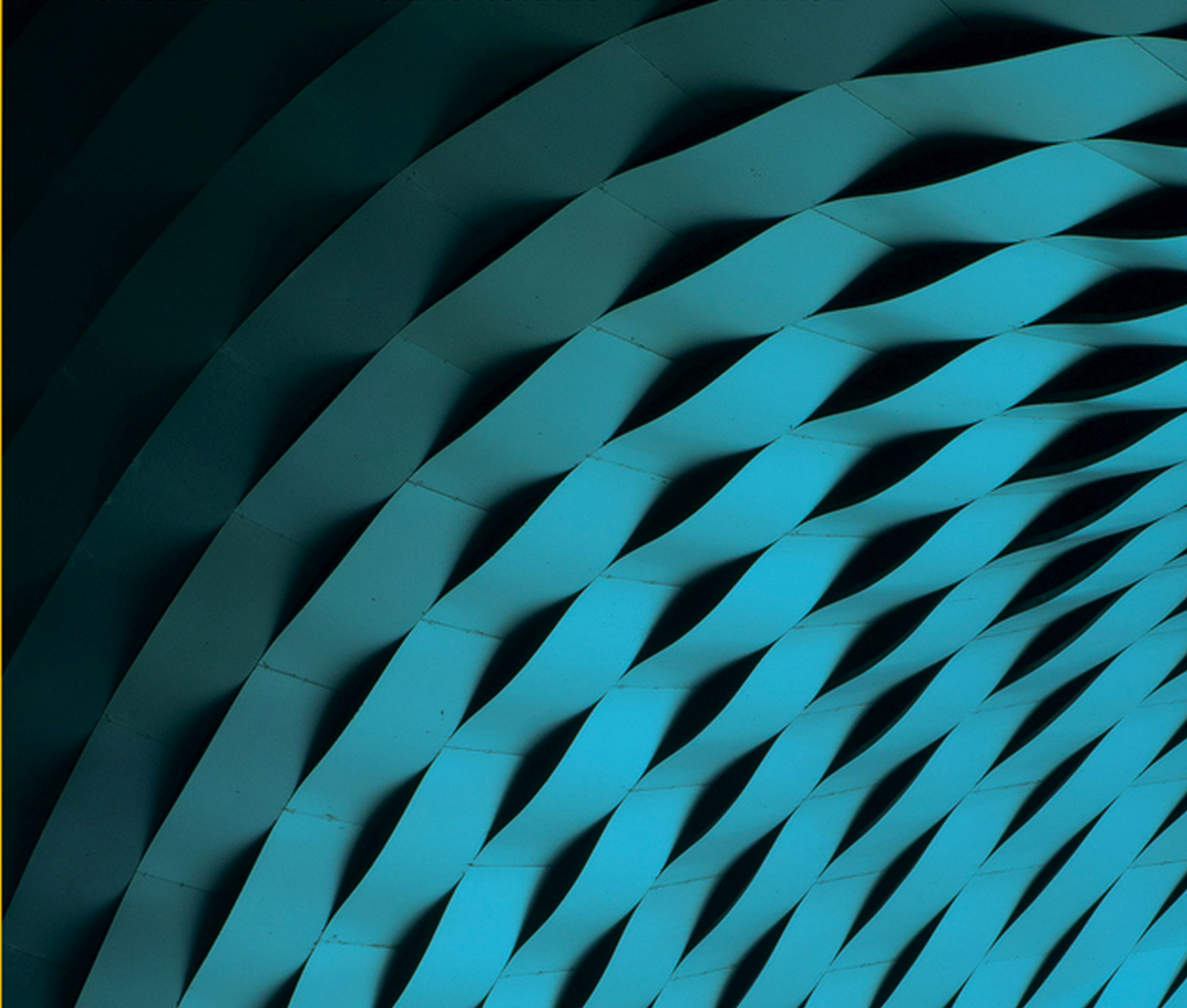
AUSTRALIA AND NEW ZEALAND EDITION

APPLICATIONS VERSION

ELEMENTARY LINEAR ALGEBRA

HOWARD ANTON • CHRIS RORRES • ANTON KAUL

TWELFTH EDITION



WILEY

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10 9 8 7 6 5 4 3 2 1

To
My wife, Pat

My children, Brian, David, and Lauren

My parents, Shirley and Benjamin

In memory of Prof. Leon Bahar,
who fostered my love of mathematics

My benefactor, Stephen Girard (1750–1831),
whose philanthropy changed my life

Howard Anton

To
Billie

Chris Rorres

To
My wife, Michelle, and my boys, Ulysses and Seth

Anton Kaul

About the Authors

HOWARD ANTON obtained his B.A. from Lehigh University, his M.A. from the University of Illinois, and his Ph.D. from the Polytechnic Institute of Brooklyn (now part of New York University), all in mathematics. In the early 1960s he was employed by the Burroughs Corporation at Cape Canaveral, Florida, where he worked on mathematical problems in the manned space program. In 1968 he joined the Mathematics Department of Drexel University, where he taught and did research until 1983. Since then he has devoted the majority of his time to textbook writing and activities for mathematical associations. Dr. Anton was president of the Eastern Pennsylvania and Delaware Section of the Mathematical Association of America, served on the Board of Governors of that organization, and guided the creation of its Student Chapters. He is the coauthor of a popular calculus text and has authored numerous research papers in functional analysis, topology, and approximation theory. His textbooks are among the most widely used in the world. There are now more than 200 versions of his books, including translations into Spanish, Arabic, Portuguese, Italian, Indonesian, French, and Japanese. For relaxation, Dr. Anton enjoys travel and photography. This text is the recipient of the *Textbook Excellence Award* by Textbook & Academic Authors Association.

CHRIS RORRES earned his B.S. degree from Drexel University and his Ph.D. from the Courant Institute of New York University. He was a faculty member of the Department of Mathematics at Drexel University for more than 30 years where, in addition to teaching, he published research articles in solar engineering, acoustic scattering, population dynamics, computer system reliability, geometry of archaeological sites, optimal animal harvesting policies, and decision theory. He retired from Drexel in 2001 as a Professor Emeritus of Mathematics, and then continued in a research position at the School of Veterinary Medicine at the University of Pennsylvania for fourteen years where he did mathematical modeling of animal epidemics. In addition to writing, he is now doing mathematical consulting.

Dr. Rorres is a recognized expert on the life and work of Archimedes. His highly acclaimed website, *Archimedes Home Page*, is a virtual book that has become an important teaching tool for students around the world. Dr. Rorres has been featured in several television documentaries (BBC and *NOVA*) about Archimedes. In 2013 Dr. Rorres organized a conference—ARCHIMEDES IN THE 21st CENTURY: *A World Conference at the Courant Institute of Mathematical Sciences*—and edited its proceedings, which were published in 2017.

ANTON KAUL received his B.S. from UC Davis and his M.S. and Ph.D. from Oregon State University. He held positions at the University of South Florida and Tufts University before joining the faculty at Cal Poly, San Luis Obispo in 2003, where he is currently a professor in the Mathematics Department. In addition to his work on mathematics textbooks, Dr. Kaul has done research in the area of geometric group theory and has published journal articles on Coxeter groups and their automorphisms. He is also an avid baseball fan and old-time banjo player.

Preface

We are proud that this book is the recipient of the *Textbook Excellence Award* from the Text & Academic Authors Association. Its quality owes much to the many professors who have taken the time to write and share their pedagogical expertise. We thank them all.

This 12th edition of *Elementary Linear Algebra, Applications Version* has a new contemporary design, many new exercises, a new application on the mathematics of facial recognition, and some organizational changes suggested by the classroom experience of many users. However, the fundamental philosophy of this book has not changed. It provides an introductory treatment of linear algebra that is suitable for a first undergraduate course. Its aim is to present the fundamentals of the subject in the clearest possible way, with sound pedagogy being the main consideration. Although calculus is not a prerequisite, some optional material here is clearly marked for students with a calculus background. If desired, that material can be omitted without loss of continuity. Technology is not required to use this text. However, clearly marked exercises that require technology are included for those who would like to use MATLAB, Mathematica, Maple, or other software with linear algebra capabilities. Supporting data files are posted on both of the following sites:

www.howardanton.com
www.wiley.com/college/anton

Summary of Changes in this Edition

Many parts of the text have been revised based on an extensive set of reviews. Here are the primary changes:

- **New Application Section** — A new section on the mathematics of facial recognition has been added to Chapter 10.
- **Earlier Linear Transformations** — Selected material on linear transformations that was covered later in the previous edition has been moved to Chapter 1 to provide a more complete early introduction to the topic. Specifically, some of the material in Sections 4.10 and 4.11 of the previous edition was extracted to form the new Section 1.9, and the remaining material is now in Section 8.6.
- **New Section 4.3 Devoted to Spanning Sets** — Section 4.2 of the previous edition dealt with both subspaces and spanning sets. Classroom experience has suggested that too many concepts were being introduced at once, so we have slowed down the pace and split off the material on spanning sets to create a new Section 4.3.
- **New Examples** — New examples have been added, where needed, to support the exercise sets.
- **New Exercises** — New exercises have been added with special attention to the expanded early introduction to linear transformations.

Hallmark Features

- **Interrelationships Among Concepts** — One of our main pedagogical goals is to convey to the student that linear algebra is not a collection of isolated definitions and techniques, but is rather a cohesive subject with interrelated ideas. One way in which we do this is by using a crescendo of theorems labeled "Equivalent Statements" that continually revisit relationships among systems of equations, matrices, determinants, vectors, linear transformations, and eigenvalues. To get a general sense of this pedagogical technique see Theorems 1.5.3, 1.6.4, 2.3.8, 4.9.8, 5.1.5, 6.4.5, and 8.2.4.
- **Smooth Transition to Abstraction** — Because the transition from Euclidean spaces to general vector spaces is difficult for many students, considerable effort is devoted to explaining the purpose of abstraction and helping the student to "visualize" abstract ideas by drawing analogies to familiar geometric ideas.
- **Mathematical Precision** — We try to be as mathematically precise as is reasonable for students at this level. But we recognize that mathematical precision is something to be learned, so proofs are presented in a patient style that is tailored for beginners.
- **Suitability for a Diverse Audience** — The text is designed to serve the needs of students in engineering, computer science, biology, physics, business, and economics, as well as those majoring in mathematics.
- **Historical Notes** — We feel that it is important to give students a sense of mathematical history and to convey that real people created the mathematical theorems and equations they are studying. Accordingly, we have included numerous "Historical Notes" that put various topics in historical perspective.

About the Exercises

- **Graded Exercise Sets** — Each exercise set begins with routine drill problems and progresses to problems with more substance. These are followed by three categories of problems, the first focusing on proofs, the second on true/false exercises, and the third on problems requiring technology. This compartmentalization is designed to simplify the instructor's task of selecting exercises for homework.

- **True/False Exercises** — The true/false exercises are designed to check conceptual understanding and logical reasoning. To avoid pure guesswork, the students are required to justify their responses in some way.
- **Proof Exercises** — Linear algebra courses vary widely in their emphasis on proofs, so exercises involving proofs have been grouped for easy identification. Appendix A provides students some guidance on proving theorems.
- **Technology Exercises** — Exercises that require technology have also been grouped. To avoid burdening the student with typing, the relevant data files have been posted on the websites that accompany this text.
- **Supplementary Exercises** — Each chapter ends with a set of exercises that draws from all the sections in the chapter.

Supplementary Materials for Students Available on the Web

- **Self Testing Review** — This edition also has an exciting new supplement, called the *Linear Algebra FlashCard Review*. It is a self-study testing system based on the SQ3R study method that students can use to check their mastery of virtually every fundamental concept in this text. It is integrated into WileyPlus, and is available as a free app for iPads. The app can be obtained from the Apple Store by searching for:

  Anton Linear Algebra FlashCard Review

- **Student Solutions Manual** — This supplement provides detailed solutions to most odd-numbered exercises.
- **Maple Data Files** — Data files in Maple format for the technology exercises that are posted on the websites that accompany this text.
- **Mathematica Data Files** — Data files in Mathematica format for the technology exercises that are posted on the websites that accompany this text.
- **MATLAB Data Files** — Data files in MATLAB format for the technology exercises that are posted on the websites that accompany this text.
- **CSV Data Files** — Data files in CSV format for the technology exercises that are posted on the websites that accompany this text.
- **How to Read and Do Proofs** — A series of videos created by Prof. Daniel Solow of the Weatherhead School of Management, Case Western Reserve University, that present various strategies for proving theorems. These are available through WileyPLUS as well as the websites listed previously. There is also a guide for locating the appropriate videos for specific proofs in the text.
- **MATLAB Linear Algebra Manual and Laboratory Projects** — This supplement contains a set of laboratory projects written by Prof. Dan Seth of West Texas A&M University. It is designed to help students learn key linear algebra concepts by using MATLAB and is available in PDF form without charge to students at schools adopting the 12th edition of this text.
- **Data Files** — The data files needed for the MATLAB Linear Algebra Manual and Lab Projects supplement.
- **How to Open and Use MATLAB Files** — Instructional document on how to download, open, and use the MATLAB files accompanying this text.

Supplementary Materials for Instructors

- **Instructor Solutions Manual** — This supplement provides worked-out solutions to most exercises in the text.
- **PowerPoint Slides** — A series of slides that display important definitions, examples, graphics, and theorems in the book. These can also be distributed to students as review materials or to simplify note-taking.
- **Test Bank** — Test questions and sample examinations in PDF or LaTeX form.
- **Image Gallery** — Digital repository of images from the text that instructors may use to generate their own PowerPoint slides.
- **WileyPLUS** — An online environment for effective teaching and learning. WileyPLUS builds student confidence by taking the guesswork out of studying and by providing a clear roadmap of what to do, how to do it, and whether it was done right. Its purpose is to motivate and foster initiative so instructors can have a greater impact on classroom achievement and beyond.
- **WileyPLUS Question Index** — This document lists every question in the current WileyPLUS course and provides the name, associated learning objective, question type, and difficulty level for each. If available, it also shows the correlation between the previous edition WileyPLUS question and the current WileyPLUS question, so instructors can conveniently see the evolution of a question and reuse it from previous semester assignments.

A Guide for the Instructor

Although linear algebra courses vary widely in content and philosophy, most courses fall into two categories, those with roughly 40 lectures, and those with roughly 30 lectures. Accordingly, we have created the following long and short templates as possible starting points for constructing your own course outline. Keep in mind that these are just guides, and we fully expect that you will want to customize them to fit your own interests and requirements. Neither of these sample templates includes applications, so keep that in mind as you work with them.

	Long Template	Short Template
Chapter 1: Systems of Linear Equations and Matrices	8 lectures	6 lectures
Chapter 2: Determinants	3 lectures	3 lectures
Chapter 3: Euclidean Vector Spaces	4 lectures	3 lectures
Chapter 4: General Vector Spaces	8 lectures	7 lectures
Chapter 5: Eigenvalues and Eigenvectors	3 lectures	3 lectures
Chapter 6: Inner Product Spaces	3 lectures	2 lectures
Chapter 7: Diagonalization and Quadratic Forms	4 lectures	3 lectures
Chapter 8: General Linear Transformations	4 lectures	2 lectures
Chapter 9: Numerical Methods	2 lectures	1 lecture
Chapter 10: Applications of Linear Algebra	As Time Permits	
Total:	**39 lectures**	**30 lectures**

Reviewers

The following people reviewed the plans for this edition, critiqued much of the content, and provided insightful pedagogical advice:

Charles Ekene Chika, *University of Texas at Dallas*

Marian Hukle, *University of Kansas*

Bin Jiang, *Portland State University*

Mike Panahi, *El Centro College*

Christopher Rasmussen, *Wesleyan University*

Nathan Reff, *The College at Brockport: SUNY*

Mark Smith, *Miami University*

Rebecca Swanson, *Colorado School of Mines*

R. Scott Williams, *University of Central Oklahoma*

Pablo Zafra, *Kean University*

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Our deep appreciation is due to a number of people who have contributed to this edition in many ways:

Prof. Mark Smith, who critiqued the FlashCard program and suggested valuable improvements to the text exposition.

Prof. Derek Hein, whose keen eye helped us to correct some subtle inaccuracies.

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HOWARD ANTON
CHRIS RORRES
ANTON KAUL

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CHAPTER 1

Systems of Linear Equations and Matrices

CHAPTER CONTENTS

Introduction

Information in science, business, and mathematics is often organized into rows and columns to form rectangular arrays called "matrices" (plural of "matrix"). Matrices often appear as tables of numerical data that arise from physical observations, but they occur in various mathematical contexts as well. For example, we will see in this chapter that all of the information required to solve a system of equations such as

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

and that the solution of the system can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs for solving systems of equations because computers are well suited for manipulating arrays of numerical information. However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a multitude of practical applications. It is the study of matrices and related topics that forms the mathematical field that we call "linear algebra." In this chapter we will begin our study of matrices.

1.1 Introduction to Systems of Linear Equations

Systems of linear equations and their solutions constitute one of the major topics that we will study in this course. In this first section we will introduce some basic terminology and discuss a method for solving such systems.

Linear Equations

Recall that in two dimensions a line in a rectangular xy-coordinate system can be represented by an equation of the form

$$ax + by = c \quad (a, b \text{ not both } 0)$$

and in three dimensions a plane in a rectangular xyz-coordinate system can be represented by an equation of the form

$$ax + by + cz = d \quad (a, b, c \text{ not all } 0)$$

These are examples of "linear equations," the first being a linear equation in the variables x and y and the second a linear equation in the variables x, y, and z. More generally, we define a ***linear equation*** in the n variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, a_2, \ldots, a_n$ and b are constants, and the a's are not all zero. In the special cases where $n = 2$ or $n = 3$, we will often use variables without subscripts and write linear equations as

$$a_1x + a_2y = b \tag{2}$$

$$a_1x + a_2y + a_3z = b \tag{3}$$

In the special case where $b = 0$, Equation (1) has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \tag{4}$$

which is called a ***homogeneous linear equation*** in the variables $x_1, x_2, \ldots, x_n$.

EXAMPLE 1 | Linear Equations

Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions. The following are linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

The following are not linear equations:

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$

$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

A finite set of linear equations is called a ***system of linear equations*** or, more briefly, a ***linear system***. The variables are called ***unknowns***. For example, system (5) that follows has unknowns x and y, and system (6) has unknowns x_1, x_2, and x_3.

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned} \qquad \begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned} \tag{5–6}$$

A general linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{7}$$

The double subscripting on the coefficients a_{ij} of the unknowns gives their location in the system—the first subscript indicates the equation in which the coefficient occurs, and the second indicates which unknown it multiplies. Thus, a_{12} is in the first equation and multiplies x_2.

A ***solution*** of a linear system in n unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

makes each equation a true statement. For example, the system in (5) has the solution

$$x = 1, \quad y = -2$$

and the system in (6) has the solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$

These solutions can be written more succinctly as

$$(1, -2) \quad \text{and} \quad (1, 2, -1)$$

in which the names of the variables are omitted. This notation allows us to interpret these solutions geometrically as points in two-dimensional and three-dimensional space. More generally, a solution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

of a linear system in n unknowns can be written as

$$(s_1, s_2, \ldots, s_n)$$

which is called an ***ordered n-tuple***. With this notation it is understood that all variables appear in the same order in each equation. If $n = 2$, then the n-tuple is called an ***ordered pair***, and if $n = 3$, then it is called an ***ordered triple***.

Linear Systems in Two and Three Unknowns

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the xy-plane. Each solution (x, y) of this system corresponds to a point of intersection of the lines, so there are three possibilities (**Figure 1.1.1**):

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

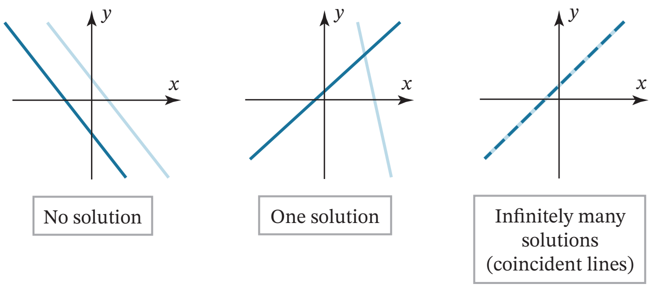


FIGURE 1.1.1

In general, we say that a linear system is ***consistent*** if it has at least one solution and ***inconsistent*** if it has no solutions. Thus, a *consistent* linear system of two equations in two unknowns has either one solution or infinitely many solutions—there are no other possibilities. The same is true for a linear system of three equations in three unknowns

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities—no solutions, one solution, or infinitely many solutions (**Figure 1.1.2**).

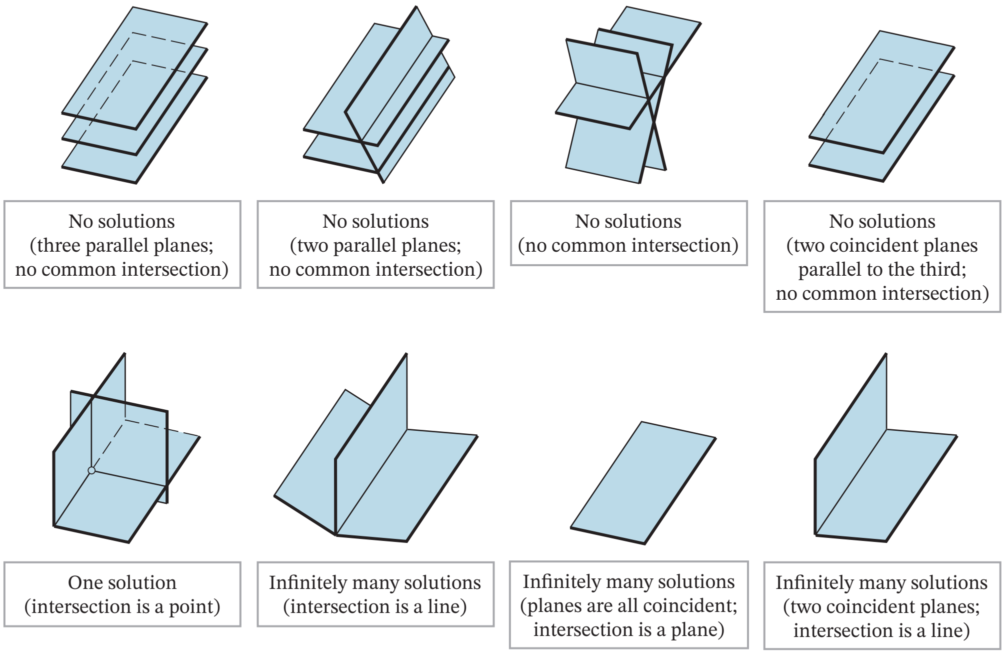


FIGURE 1.1.2

We will prove later that our observations about the number of solutions of linear systems of two equations in two unknowns and linear systems of three equations in three unknowns actually hold for *all* linear systems. That is:

> *Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

EXAMPLE 2 | A Linear System with One Solution

Solve the linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the simplified system

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

From the second equation we obtain $y = \frac{4}{3}$, and on substituting this value in the first equation we obtain $x = 1 + y = \frac{7}{3}$. Thus, the system has the unique solution

$$x = \tfrac{7}{3}, \quad y = \tfrac{4}{3}$$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3}, \frac{4}{3}\right)$. We leave it for you to check this by graphing the lines.

EXAMPLE 3 | A Linear System with No Solutions

Solve the linear system

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -3 times the first equation to the second equation. This yields the simplified system

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

The second equation is contradictory, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. We leave it for you to check this by graphing the lines or by showing that they have the same slope but different y-intercepts.

EXAMPLE 4 | A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -4 times the first equation to the second. This yields the simplified system

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restrictions on x and y and hence can be omitted. Thus, the solutions of the system are those values of x and y that satisfy the single equation

$$4x - 2y = 1 \tag{8}$$

Geometrically, this means the lines corresponding to the two equations in the original system coincide. One way to describe the solution set is to solve this equation for x in terms of y to

In Example 4 we could have also obtained parametric equations for the solutions by solving (8) for y in terms of x and letting $x = t$ be the parameter. The resulting parametric equations would look different but would define the same solution set.

obtain $x = \frac{1}{4} + \frac{1}{2}y$ and then assign an arbitrary value t (called a ***parameter***) to y. This allows us to express the solution by the pair of equations (called ***parametric equations***)

$$x = \tfrac{1}{4} + \tfrac{1}{2}t, \quad y = t$$

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter t. For example, $t = 0$ yields the solution $\left(\frac{1}{4}, 0\right)$, $t = 1$ yields the solution $\left(\frac{3}{4}, 1\right)$, and $t = -1$ yields the solution $\left(-\frac{1}{4}, -1\right)$. You can confirm that these are solutions by substituting their coordinates into the given equations.

EXAMPLE 5 | A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of x, y, and z that satisfy the equation

$$x - y + 2z = 5 \tag{9}$$

automatically satisfy all three equations. Thus, it suffices to find the solutions of (9). We can do this by first solving this equation for x in terms of y and z, then assigning arbitrary values r and s (parameters) to these two variables, and then expressing the solution by the three parametric equations

$$x = 5 + r - 2s, \quad y = r, \quad z = s$$

Specific solutions can be obtained by choosing numerical values for the parameters r and s. For example, taking $r = 1$ and $s = 0$ yields the solution $(6, 1, 0)$.

Augmented Matrices and Elementary Row Operations

As the number of equations and unknowns in a linear system increases, so does the complexity of the algebra involved in finding solutions. The required computations can be made more manageable by simplifying notation and standardizing procedures. For example, by mentally keeping track of the location of the +'s, the x's, and the ='s in the linear system

$$\begin{matrix} a_{11}x_1 & + a_{12}x_2 & + \cdots + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots + & a_{2n}x_n & = b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + \cdots + & a_{mn}x_n & = b_m \end{matrix}$$

we can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

As noted in the introduction to this chapter, the term "matrix" is used in mathematics to denote a rectangular array of numbers. In a later section we will study matrices in detail, but for now we will only be concerned with augmented matrices for linear systems.

This is called the ***augmented matrix*** for the system. For example, the augmented matrix for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Historical Note

Maxime Bôcher (1867–1918)

The first known use of augmented matrices appeared between 200 B.C. and 100 B.C. in a Chinese manuscript entitled *Nine Chapters of Mathematical Art*. The coefficients were arranged in columns rather than in rows, as today, but remarkably the system was solved by performing a succession of operations on the columns. The actual use of the term *augmented matrix* appears to have been introduced by the American mathematician Maxime Bôcher in his book *Introduction to Higher Algebra*, published in 1907. In addition to being an outstanding research mathematician and an expert in Latin, chemistry, philosophy, zoology, geography, meteorology, art, and music, Bôcher was an outstanding expositor of mathematics whose elementary textbooks were greatly appreciated by students and are still in demand today.

[Image: HUP Bocher, Maxime (1), olvwork650836]

The basic method for solving a linear system is to perform algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called ***elementary row operations*** on a matrix.

In the following example we will illustrate how to use elementary row operations and an augmented matrix to solve a linear system in three unknowns. Since a systematic procedure for solving linear systems will be developed in the next section, do not worry about how the steps in the example were chosen. Your objective here should be simply to understand the computations.

EXAMPLE 6 | Using Elementary Row Operations

In the left column we solve a system of linear equations by operating on the equations in the system, and in the right column we solve the same system by operating on the rows of the augmented matrix.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −2 times the first equation to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add −2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add −3 times the first equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Add −3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add −3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add −3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by −2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by −2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add −1 times the second equation to the first to obtain

$$\begin{aligned} x \quad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add −1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \qquad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution $x = 1$, $y = 2$, $z = 3$ is now evident.

The solution in this example can also be expressed as the ordered triple (1, 2, 3) with the understanding that the numbers in the triple are in the same order as the variables in the system, namely, x, y, z.

Exercise Set 1.1

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

a. $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ **b.** $x_1 + 3x_2 + x_1x_3 = 2$

c. $x_1 = -7x_2 + 3x_3$ **d.** $x_1^{-2} + x_2 + 8x_3 = 5$

e. $x_1^{3/5} - 2x_2 + x_3 = 4$ **f.** $\pi x_1 - \sqrt{2}x_2 = 7^{1/3}$

2. In each part, determine whether the equation is linear in x and y.

a. $2^{1/3}x + \sqrt{3}y = 1$ **b.** $2x^{1/3} + 3\sqrt{y} = 1$

c. $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$ **d.** $\frac{\pi}{7}\cos x - 4y = 0$

e. $xy = 1$ **f.** $y + 7 = x$

3. Using the notation of Formula (7), write down a general linear system of

a. two equations in two unknowns.

b. three equations in three unknowns.

c. two equations in four unknowns.

4. Write down the augmented matrix for each of the linear systems in Exercise 3.

In each part of Exercises 5–6, find a system of linear equations in the unknowns $x_1, x_2, x_3, \ldots$, that corresponds to the given augmented matrix.

5. **a.** $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$

6. **a.** $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

In each part of Exercises 7–8, find the augmented matrix for the linear system.

7. **a.** $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$ **b.** $\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$

c. $\begin{aligned} 2x_2 - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$

8. **a.** $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$ **b.** $\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$

c. $\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$

9. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

a. $(3, 1, 1)$ **b.** $(3, -1, 1)$ **c.** $(13, 5, 2)$

d. $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$ **e.** $(17, 7, 5)$

10. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} x + 2y - 2z &= 3 \\ 3x - y + z &= 1 \\ -x + 5y - 5z &= 5 \end{aligned}$$

a. $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$ **b.** $\left(\frac{5}{7}, \frac{8}{7}, 0\right)$ **c.** $(5, 8, 1)$

d. $\left(\frac{5}{7}, \frac{10}{7}, \frac{2}{7}\right)$ **e.** $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$

11. In each part, solve the linear system, if possible, and use the result to determine whether the lines represented by the equations in the system have zero, one, or infinitely many points of intersection. If there is a single point of intersection, give its coordinates, and if there are infinitely many, find parametric equations for them.

a. $\begin{aligned} 3x - 2y &= 4 \\ 6x - 4y &= 9 \end{aligned}$ **b.** $\begin{aligned} 2x - 4y &= 1 \\ 4x - 8y &= 2 \end{aligned}$ **c.** $\begin{aligned} x - 2y &= 0 \\ x - 4y &= 8 \end{aligned}$

12. Under what conditions on a and b will the linear system have no solutions, one solution, infinitely many solutions?

$$\begin{aligned} 2x - 3y &= a \\ 4x - 6y &= b \end{aligned}$$

In each part of Exercises 13–14, use parametric equations to describe the solution set of the linear equation.

13. **a.** $7x - 5y = 3$

b. $3x_1 - 5x_2 + 4x_3 = 7$

c. $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

d. $3v - 8w + 2x - y + 4z = 0$

14. **a.** $x + 10y = 2$

b. $x_1 + 3x_2 - 12x_3 = 3$

c. $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

d. $v + w + x - 5y + 7z = 0$

In Exercises 15–16, each linear system has infinitely many solutions. Use parametric equations to describe its solution set.

15. **a.** $\begin{aligned} 2x - 3y &= 1 \\ 6x - 9y &= 3 \end{aligned}$

b. $\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 3x_1 + 9x_2 - 3x_3 &= -12 \\ -x_1 - 3x_2 + x_3 &= 4 \end{aligned}$

16. **a.** $\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$ **b.** $\begin{aligned} 2x - y + 2z &= -4 \\ 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$

In Exercises 17–18, find a single elementary row operation that will create a 1 in the upper left corner of the given augmented matrix and will not create any fractions in its first row.

17. **a.** $\begin{bmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & -1 & -5 & 0 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

18. **a.** $\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$ **b.** $\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$

In Exercises 19–20, find all values of k for which the given augmented matrix corresponds to a consistent linear system.

19. **a.** $\begin{bmatrix} 1 & k & -4 \\ 4 & 8 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & k & -1 \\ 4 & 8 & -4 \end{bmatrix}$

20. **a.** $\begin{bmatrix} 3 & -4 & k \\ -6 & 8 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} k & 1 & -2 \\ 4 & -1 & 2 \end{bmatrix}$

21. The curve $y = ax^2 + bx + c$ shown in the accompanying figure passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that the coefficients a, b, and c form a solution of the system of linear equations whose augmented matrix is

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$

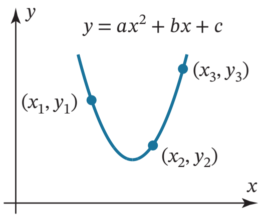


FIGURE Ex-21

22. Explain why each of the three elementary row operations does not affect the solution set of a linear system.

23. Show that if the linear equations

$$x_1 + kx_2 = c \qquad \text{and} \qquad x_1 + lx_2 = d$$

have the same solution set, then the two equations are identical (i.e., $k = l$ and $c = d$).

24. Consider the system of equations

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \\ ex + fy &= m \end{aligned}$$

Discuss the relative positions of the lines $ax + by = k$, $cx + dy = l$, and $ex + fy = m$ when

a. the system has no solutions.

b. the system has exactly one solution.

c. the system has infinitely many solutions.

25. Suppose that a certain diet calls for 7 units of fat, 9 units of protein, and 16 units of carbohydrates for the main meal, and suppose that an individual has three possible foods to choose from to meet these requirements:

Food 1: Each ounce contains 2 units of fat, 2 units of protein, and 4 units of carbohydrates.

Food 2: Each ounce contains 3 units of fat, 1 unit of protein, and 2 units of carbohydrates.

Food 3: Each ounce contains 1 unit of fat, 3 units of protein, and 5 units of carbohydrates.

Let x, y, and z denote the number of ounces of the first, second, and third foods that the dieter will consume at the main meal. Find (but do not solve) a linear system in x, y, and z whose solution tells how many ounces of each food must be consumed to meet the diet requirements.

26. Suppose that you want to find values for a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the points $(1, 1)$, $(2, 4)$, and $(-1, 1)$. Find (but do not solve) a system of linear equations whose solutions provide values for a, b, and c. How many solutions would you expect this system of equations to have, and why?

27. Suppose you are asked to find three real numbers such that the sum of the numbers is 12, the sum of two times the first plus the second plus two times the third is 5, and the third number is one more than the first. Find (but do not solve) a linear system whose equations describe the three conditions.

True-False Exercises

TF. In parts **(a)–(h)** determine whether the statement is true or false, and justify your answer.

a. A linear system whose equations are all homogeneous must be consistent.

b. Multiplying a row of an augmented matrix through by zero is an acceptable elementary row operation.

c. The linear system

$$\begin{aligned} x - y &= 3 \\ 2x - 2y &= k \end{aligned}$$

cannot have a unique solution, regardless of the value of k.

d. A single linear equation with two or more unknowns must have infinitely many solutions.

e. If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent.

f. If each equation in a consistent linear system is multiplied through by a constant c, then all solutions to the new system can be obtained by multiplying solutions from the original system by c.

g. Elementary row operations permit one row of an augmented matrix to be subtracted from another.

h. The linear system with corresponding augmented matrix

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

is consistent.

Working with Technology

T1. Solve the linear systems in Examples 2, 3, and 4 to see how your technology utility handles the three types of systems.

T2. Use the result in Exercise 21 to find values of a, b, and c for which the curve $y = ax^2 + bx + c$ passes through the points $(-1, 1, 4)$, $(0, 0, 8)$, and $(1, 1, 7)$.

1.2 Gaussian Elimination

In this section we will develop a systematic procedure for solving systems of linear equations. The procedure is based on the idea of performing certain operations on the rows of the augmented matrix that simplify it to a form from which the solution of the system can be ascertained by inspection.

Considerations in Solving Linear Systems

When considering methods for solving systems of linear equations, it is important to distinguish between large systems that must be solved by computer and small systems that can be solved by hand. For example, there are many applications that lead to linear systems in thousands or even millions of unknowns. Large systems require special techniques to deal with issues of memory size, roundoff errors, solution time, and so forth. Such techniques are studied in the field of ***numerical analysis*** and will only be touched on in this text. However, almost all of the methods that are used for large systems are based on the ideas that we will develop in this section.

Echelon Forms

In Example 6 of the last section, we solved a linear system in the unknowns x, y, and z by reducing the augmented matrix to the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

from which the solution $x = 1$, $y = 2$, $z = 3$ became evident. This is an example of a matrix that is in ***reduced row echelon form***. To be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row echelon form***. (Thus, a matrix in reduced row echelon form is of necessity in row echelon form, but not conversely.)

EXAMPLE 1 | Row Echelon and Reduced Row Echelon Form

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 2 | More on Row Echelon and Reduced Row Echelon Form

As Example 1 illustrates, a matrix in row echelon form has zeros below each leading 1, whereas a matrix in reduced row echelon form has zeros below *and above* each leading 1. Thus, with any real numbers substituted for the $*$'s, all matrices of the following types are in row echelon form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

All matrices of the following types are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

If, by a sequence of elementary row operations, the augmented matrix for a system of linear equations is put in *reduced* row echelon form, then the solution set can be obtained either by inspection or by converting certain linear equations to parametric form. Here are some examples.

EXAMPLE 3 | Unique Solution

Suppose that the augmented matrix for a linear system in the unknowns x_1, x_2, x_3, and x_4 has been reduced by elementary row operations to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

This matrix is in reduced row echelon form and corresponds to the equations

$$\begin{aligned} x_1 \qquad\qquad\qquad &= 3 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= 0 \\ x_4 &= 5 \end{aligned}$$

Thus, the system has a unique solution, namely, $x_1 = 3, x_2 = -1, x_3 = 0, x_4 = 5$, which can also be expressed as the 4-tuple $(3, -1, 0, 5)$.

EXAMPLE 4 | Linear Systems in Three Unknowns

In each part, suppose that the augmented matrix for a linear system in the unknowns x, y, and z has been reduced by elementary row operations to the given reduced row echelon form. Solve the system.

$$(a)\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (c)\ \begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution (*a*) The equation that corresponds to the last row of the augmented matrix is

$$0x + 0y + 0z = 1$$

Since this equation is not satisfied by any values of x, y, and z, the system is inconsistent.

Solution (*b*) The equation that corresponds to the last row of the augmented matrix is

$$0x + 0y + 0z = 0$$

This equation can be omitted since it imposes no restrictions on x, y, and z; hence, the linear system corresponding to the augmented matrix is

$$\begin{aligned} x \quad\ + 3z &= -1 \\ y - 4z &= \ \ 2 \end{aligned}$$

In general, the variables in a linear system that correspond to the leading 1's in its augmented matrix are called the ***leading variables***, and the remaining variables are called the ***free variables***. In this case the leading variables are x and y, and the variable z is the only free variable. Solving for the leading variables in terms of the free variables gives

$$\begin{aligned} x &= -1 - 3z \\ y &= 2 + 4z \end{aligned}$$

From these equations we see that the free variable z can be treated as a parameter and assigned an arbitrary value t, which then determines values for x and y. Thus, the solution set can be represented by the parametric equations

$$x = -1 - 3t, \quad y = 2 + 4t, \quad z = t$$

By substituting various values for t in these equations we can obtain various solutions of the system. For example, setting $t = 0$ yields the solution

$$x = -1, \quad y = 2, \quad z = 0$$

and setting $t = 1$ yields the solution

$$x = -4, \quad y = 6, \quad z = 1$$

Solution (*c*) As explained in part (b), we can omit the equations corresponding to the zero rows, in which case the linear system associated with the augmented matrix consists of the single equation

$$x - 5y + z = 4 \tag{1}$$

from which we see that the solution set is a plane in three-dimensional space. Although (1) is a valid form of the solution set, there are many applications in which it is preferable to express the solution set in parametric form. We can convert (1) to parametric form by solving for the leading variable x in terms of the free variables y and z to obtain

$$x = 4 + 5y - z$$

From this equation we see that the free variables can be assigned arbitrary values, say $y = s$ and $z = t$, which then determine the value of x. Thus, the solution set can be expressed parametrically as

$$x = 4 + 5s - t, \quad y = s, \quad z = t \tag{2}$$

We will usually denote parameters in a general solution by the letters $r, s, t, \ldots$, but any letters that do not conflict with the names of the unknowns can be used. For systems with more than three unknowns, subscripted letters such as $t_1, t_2, t_3, \ldots$ are convenient.

Formulas, such as (2), that express the solution set of a linear system parametrically have some associated terminology.

Definition 1

If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called a ***general solution*** of the system.

Thus, for example, Formula (2) is a general solution of system (iii) in the previous example.

Elimination Methods

We have just seen how easy it is to solve a system of linear equations once its augmented matrix is in reduced row echelon form. Now we will give a step-by-step *algorithm* that can be used to reduce any matrix to reduced row echelon form. As we state each step in the algorithm, we will illustrate the idea by reducing the following matrix to reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

← The first and second rows in the preceding matrix were interchanged.

Step 3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

← The first row of the preceding matrix was multiplied by $\frac{1}{2}$.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

← -2 times the first row of the preceding matrix was added to the third row.

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the *entire* matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

The first row in the submatrix was multiplied by $-\frac{1}{2}$ to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

−5 times the first row of the submatrix was added to the second row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again to Step 1.

Leftmost nonzero column in the new submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submatrix was multiplied by 2 to introduce a leading 1.

The *entire* matrix is now in row echelon form. To find the reduced row echelon form we need the following additional step.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\frac{7}{2}$ times the third row of the preceding matrix was added to the second row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

−6 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the second row was added to the first row.

The last matrix is in reduced row echelon form.

The algorithm we have just described for reducing a matrix to reduced row echelon form is called ***Gauss–Jordan elimination***. It consists of two parts, a ***forward phase*** in which zeros are introduced below the leading 1's and a ***backward phase*** in which zeros are introduced above the leading 1's. If only the forward phase is used, then the procedure produces a row echelon form and is called ***Gaussian elimination***. For example, in the preceding computations a row echelon form was obtained at the end of Step 5.

Historical Note

Carl Friedrich Gauss (1777–1855)

Wilhelm Jordan (1842–1899)

Although versions of Gaussian elimination were known much earlier, its importance in scientific computation became clear when the great German mathematician Carl Friedrich Gauss used it to help compute the orbit of the asteroid Ceres from limited data. What happened was this: On January 1, 1801 the Sicilian astronomer and Catholic priest Giuseppe Piazzi (1746–1826) noticed a dim celestial object that he believed might be a "missing planet." He named the object Ceres and made a limited number of positional observations but then lost the object as it neared the Sun. Gauss, then only 24 years old, undertook the problem of computing the orbit of Ceres from the limited data using a technique called "least squares," the equations of which he solved by the method that we now call "Gaussian elimination." The work of Gauss created a sensation when Ceres reappeared a year later in the constellation Virgo at almost the precise position that he predicted! The basic idea of the method was further popularized by the German engineer Wilhelm Jordan in his book on geodesy (the science of measuring Earth shapes) entitled *Handbuch der Vermessungskunde* and published in 1888.

[Images: Photo Inc/Photo Researchers/Getty Images (Gauss); https://en.wikipedia.org/wiki/Andrey_Markov#/media/File:Andrei_Markov.jpg. Public domain. (Jordan)]

EXAMPLE 5 | Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination.

$$
\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}
$$

Solution The augmented matrix for the system is

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{bmatrix}
$$

Adding -2 times the first row to the second and fourth rows gives

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{bmatrix}
$$

Multiplying the second row by -1 and then adding -5 times the new second row to the third row and -4 times the new second row to the fourth row gives

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 2
\end{bmatrix}
$$

Interchanging the third and fourth rows and then multiplying the third row of the resulting matrix by $\frac{1}{6}$ gives the row echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This completes the forward phase since there are zeros below the leading 1's.

Adding -3 times the third row to the second row and then adding 2 times the second row of the resulting matrix to the first row yields the reduced row echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This completes the backward phase since there are zeros above the leading 1's.

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned} \tag{3}$$

Note that in constructing the linear system in (3) we ignored the row of zeros in the corresponding augmented matrix. Why is this justified?

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Finally, we express the general solution of the system parametrically by assigning the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively. This yields

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \tfrac{1}{3}$$

Homogeneous Linear Systems

A system of linear equations is said to be ***homogeneous*** if the constant terms are all zero; that is, the system has the form

$$\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = 0 \end{matrix}$$

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution. This solution is called the ***trivial solution***; if there are other solutions, they are called ***nontrivial solutions***.

Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

In the special case of a homogeneous linear system of two equations in two unknowns, say

$$\begin{aligned} a_1x + b_1y &= 0 \quad [a_1, b_1 \text{ not both zero}] \\ a_2x + b_2y &= 0 \quad [a_2, b_2 \text{ not both zero}] \end{aligned}$$

the graphs of the equations are lines through the origin, and the trivial solution corresponds to the point of intersection at the origin (**Figure 1.2.1**).

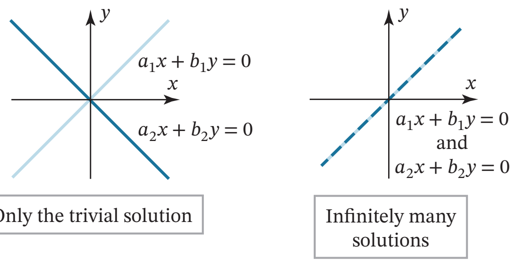


FIGURE 1.2.1

There is one case in which a homogeneous system is assured of having nontrivial solutions—namely, whenever the system involves more unknowns than equations. To see why, consider the following example of four equations in six unknowns.

EXAMPLE 6 | A Homogeneous System

Use Gauss–Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned} \tag{4}$$

Solution Observe that this system is the same as that in Example 5 except for the constants on the right side, which in this case are all zero. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix} \tag{5}$$

which is the same as that in Example 5 except for the entries in the last column, which are all zeros in this case. Thus, the reduced row echelon form of this matrix will be the same as that of the augmented matrix in Example 5, except for the last column. However, a moment's reflection will make it evident that a column of zeros is not changed by an elementary row operation, so the reduced row echelon form of (5) is

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad\qquad &= 0 \\ x_6 &= 0 \end{aligned}$$

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned} \tag{7}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, then we can express the solution set parametrically as

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

Note that the trivial solution results when $r = s = t = 0$.

Free Variables in Homogeneous Linear Systems

Example 6 illustrates two important points about solving homogeneous linear systems:

1. Elementary row operations do not alter columns of zeros in a matrix, so the reduced row echelon form of the augmented matrix for a homogeneous linear system has a final column of zeros. This implies that the linear system corresponding to the reduced row echelon form is homogeneous, just like the original system.
2. When we constructed the homogeneous linear system corresponding to augmented matrix (6), we ignored the row of zeros because the corresponding equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 = 0$$

   does not impose any conditions on the unknowns. Thus, depending on whether or not the reduced row echelon form of the augmented matrix for a homogeneous linear system has any zero rows, the linear system corresponding to that reduced row echelon form will either have the same number of equations as the original system or it will have fewer.

Now consider a general homogeneous linear system with n unknowns, and suppose that the reduced row echelon form of the augmented matrix has r nonzero rows. Since each nonzero row has a leading 1, and since each leading 1 corresponds to a leading variable, the homogeneous system corresponding to the reduced row echelon form of the augmented matrix must have r leading variables and $n - r$ free variables. Thus, this system is of the form

$$\begin{array}{llll} x_{k_1} & & & + \sum(\ ) = 0 \\ & x_{k_2} & & + \sum(\ ) = 0 \\ & & \ddots & \quad \vdots \\ & & x_{k_r} & + \sum(\ ) = 0 \end{array} \tag{8}$$

where in each equation the expression $\sum(\ )$ denotes a sum that involves the free variables, if any [see (7), for example]. In summary, we have the following result.

Theorem 1.2.1

Free Variable Theorem for Homogeneous Systems

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem 1.2.1 has an important implication for homogeneous linear systems with more unknowns than equations. Specifically, if a homogeneous linear system has m equations in n unknowns, and if $m < n$, then it must also be true that $r < n$ (why?). This being the case, the theorem implies that there is at least one free variable, and this implies that the system has infinitely many solutions. Thus, we have the following result.

Note that Theorem 1.2.2 applies only to homogeneous systems—a *nonhomogeneous* system with more unknowns than equations need not be consistent. However, we will prove later that if a nonhomogeneous system with more unknowns than equations is consistent, then it has infinitely many solutions.

Theorem 1.2.2

A homogeneous linear system with more unknowns than equations has infinitely many solutions.

In retrospect, we could have anticipated that the homogeneous system in Example 6 would have infinitely many solutions since it has four equations in six unknowns.

Gaussian Elimination and Back-Substitution

For small linear systems that are solved by hand (such as most of those in this text), Gauss–Jordan elimination (reduction to reduced row echelon form) is a good procedure to use. However, for large linear systems that require a computer solution, it is generally more efficient to use Gaussian elimination (reduction to row echelon form) followed by a technique known as ***back-substitution*** to complete the process of solving the system. The next example illustrates this technique.

EXAMPLE 7 | Example 5 Solved by Back-Substitution

From the computations in Example 5, a row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To solve the corresponding system of equations

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad + 3x_6 &= 1 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

we proceed as follows:

Step 1. Solve the equations for the leading variables.

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Step 2. Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.

Substituting $x_6 = \frac{1}{3}$ into the second equation yields

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Substituting $x_3 = -2x_4$ into the first equation yields

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Step 3. Assign arbitrary values to the free variables, if any.

If we now assign x_2, x_4, and x_5 the arbitrary values r, s, and t, respectively, the general solution is given by the formulas

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \tfrac{1}{3}$$

This agrees with the solution obtained in Example 5.

EXAMPLE 8 | Existence and Uniqueness of Solutions

Suppose that the matrices below are augmented matrices for linear systems in the unknowns x_1, x_2, x_3, and x_4. These matrices are all in row echelon form but not reduced row echelon form. Discuss the existence and uniqueness of solutions to the corresponding linear systems

$$(a)\ \begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(c)\ \begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution (*a*) The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

from which it is evident that the system is inconsistent.

Solution (*b*) The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

which has no effect on the solution set. In the remaining three equations the variables x_1, x_2, and x_3 correspond to leading 1's and hence are leading variables. The variable x_4 is a free variable. With a little algebra, the leading variables can be expressed in terms of the free variable, and the free variable can be assigned an arbitrary value. Thus, the system must have infinitely many solutions.

Solution (*c*) The last row corresponds to the equation

$$x_4 = 0$$

which gives us a numerical value for x_4. If we substitute this value into the third equation, namely,

$$x_3 + 6x_4 = 9$$

we obtain $x_3 = 9$. You should now be able to see that if we continue this process and substitute the known values of x_3 and x_4 into the equation corresponding to the second row, we will obtain a unique numerical value for x_2; and if, finally, we substitute the known values of x_4, x_3, and x_2 into the equation corresponding to the first row, we will produce a unique numerical value for x_1. Thus, the system has a unique solution.

Some Facts About Echelon Forms

There are three facts about row echelon forms and reduced row echelon forms that are important to know but we will not prove:

1. Every matrix has a unique reduced row echelon form; that is, regardless of whether you use Gauss–Jordan elimination or some other sequence of elementary row operations, the same reduced row echelon form will result in the end.*
2. Row echelon forms are not unique; that is, different sequences of elementary row operations can result in different row echelon forms.

*A proof of this result can be found in the article "The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof," by Thomas Yuster, *Mathematics Magazine*, Vol. 57, No. 2, 1984, pp. 93–94.

3. Although row echelon forms are not unique, the reduced row echelon form and all row echelon forms of a matrix A have the same number of zero rows, and the leading 1's always occur in the same positions. Those are called the ***pivot positions*** of A. The columns containing the leading 1's in a row echelon or reduced row echelon form of A are called the ***pivot columns*** of A, and the rows containing the leading 1's are called the ***pivot rows*** of A. A *nonzero* entry in a pivot position of A is called a ***pivot*** of A.

EXAMPLE 9 | Pivot Positions and Columns

Earlier in this section (immediately after Definition 1) we found a row echelon form of

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \quad \text{to be} \quad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The leading 1's occur in (row 1, column 1), (row 2, column 3), and (row 3, column 5). These are the pivot positions of A. The pivot columns of A are 1, 3, and 5, and the pivot rows are 1, 2, and 3. The pivots of A are the nonzero numbers in the pivot positions. These are marked by shaded rectangles in the following diagram.

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & \boxed{-10} & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & \boxed{-5} & -1 \end{bmatrix}$$

Pivot columns

If A is the augmented matrix for a linear system, then the pivot columns identify the leading variables. As an illustration, in Example 5 the pivot columns are 1, 3, and 6, and the leading variables are x_1, x_3, and x_6.

Roundoff Error and Instability

There is often a gap between mathematical theory and its practical implementation—Gauss–Jordan elimination and Gaussian elimination being good examples. The problem is that computers generally approximate numbers, thereby introducing ***roundoff*** errors, so unless precautions are taken, successive calculations may degrade an answer to a degree that makes it useless. Algorithms in which this happens are called ***unstable***. There are various techniques for minimizing roundoff error and instability. For example, it can be shown that for large linear systems Gauss–Jordan elimination involves roughly 50% more operations than Gaussian elimination, so most computer algorithms are based on the latter method. Some of these matters will be considered in Chapter 9.

Exercise Set 1.2

In Exercises 1–2, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

1. a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ **e.** $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ **g.** $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

2. a. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **e.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **g.** $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

In Exercises 3–4, suppose that the augmented matrix for a linear system has been reduced by row operations to the given row echelon form. Identify the pivot rows and columns and solve the system.

3. a. $$\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

b. $$\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

c. $$\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

d. $$\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. a. $$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

b. $$\begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$$

c. $$\begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

d. $$\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In Exercises 5–8, solve the system by Gaussian elimination.

5. $$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

6. $$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

7. $$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$$

8. $$\begin{aligned} - 2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$$

In Exercises 9–12, solve the system by Gauss–Jordan elimination.

9. Exercise 5

10. Exercise 6

11. Exercise 7

12. Exercise 8

In Exercises 13–14, determine whether the homogeneous system has nontrivial solutions by inspection (without pencil and paper).

13. $$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$$

14. $$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

In Exercises 15–22, solve the given linear system by any method.

15. $$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 \qquad &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

16. $$\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

17. $$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

18. $$\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

19. $$\begin{aligned} 2x + 2y + 4z &= 0 \\ w \qquad - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

20. $$\begin{aligned} x_1 + 3x_2 \qquad + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 \qquad &= 0 \\ - 2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

21. $$\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 \qquad - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$$

22. $$\begin{aligned} Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 - 2Z_3 \qquad - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 \qquad + Z_5 &= 0 \end{aligned}$$

In each part of Exercises 23–24, the augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer "inconclusive" if there is not enough information to make a decision.

23. a. $$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

b. $$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c. $$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d. $$\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$$

24. a. $$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

b. $$\begin{bmatrix} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{bmatrix}$$

c. $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix}$$

d. $$\begin{bmatrix} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

In Exercises 25–26, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

25. $$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

26.
$$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$$

In Exercises 27–28, what condition, if any, must a, b, and c satisfy for the linear system to be consistent?

27.
$$\begin{aligned} x + 3y - z &= a \\ x + y + 2z &= b \\ 2y - 3z &= c \end{aligned}$$

28.
$$\begin{aligned} x + 3y + z &= a \\ -x - 2y + z &= b \\ 3x + 7y - z &= c \end{aligned}$$

In Exercises 29–30, solve the following systems, where a, b, and c are constants.

29.
$$\begin{aligned} 2x + y &= a \\ 3x + 6y &= b \end{aligned}$$

30.
$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ 2x_1 + 2x_3 &= b \\ 3x_2 + 3x_3 &= c \end{aligned}$$

31. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

32. Reduce

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

to reduced row echelon form without introducing fractions at any intermediate stage.

33. Show that the following nonlinear system has 18 solutions if $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma \le 2\pi$.

$$\begin{aligned} \sin\alpha + 2\cos\beta + 3\tan\gamma &= 0 \\ 2\sin\alpha + 5\cos\beta + 3\tan\gamma &= 0 \\ -\sin\alpha - 5\cos\beta + 5\tan\gamma &= 0 \end{aligned}$$

[*Hint:* Begin by making the substitutions $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$.]

34. Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

35. Solve the following system of nonlinear equations for x, y, and z.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

[*Hint:* Begin by making the substitutions $X = x^2$, $Y = y^2$, $Z = z^2$.]

36. Solve the following system for x, y, and z.

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ -\frac{1}{x} + \frac{9}{y} + \frac{10}{z} &= 5 \end{aligned}$$

37. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.

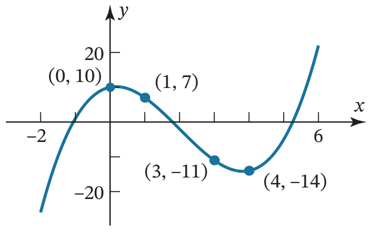


FIGURE Ex-37

38. Find the coefficients a, b, c, and d so that the circle shown in the accompanying figure is given by the equation $ax^2 + ay^2 + bx + cy + d = 0$.

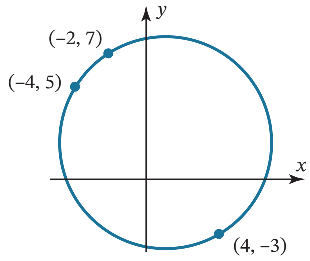


FIGURE Ex-38

39. If the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x - b_2y + c_2z &= 0 \\ a_3x + b_3y - c_3z &= 0 \end{aligned}$$

has only the trivial solution, what can be said about the solutions of the following system?

$$\begin{aligned} a_1x + b_1y + c_1z &= 3 \\ a_2x - b_2y + c_2z &= 7 \\ a_3x + b_3y - c_3z &= 11 \end{aligned}$$

40. **a.** If A is a matrix with three rows and five columns, then what is the maximum possible number of leading 1's in its reduced row echelon form?

b. If B is a matrix with three rows and six columns, then what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B?

c. If C is a matrix with five rows and three columns, then what is the minimum possible number of rows of zeros in any row echelon form of C?

41. Describe all possible reduced row echelon forms of

a. $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ **b.** $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix}$

42. Consider the system of equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \\ ex + fy &= 0 \end{aligned}$$

Discuss the relative positions of the lines $ax + by = 0$, $cx + dy = 0$, and $ex + fy = 0$ when the system has only the trivial solution and when it has nontrivial solutions.

Working with Proofs

43. **a.** Prove that if $ad - bc \neq 0$, then the reduced row echelon form of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. Use the result in part (a) to prove that if $ad - bc \neq 0$, then the linear system

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \end{aligned}$$

has exactly one solution.

True-False Exercises

TF. In parts **(a)–(i)** determine whether the statement is true or false, and justify your answer.

a. If a matrix is in reduced row echelon form, then it is also in row echelon form.

b. If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form.

c. Every matrix has a unique row echelon form.

d. A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.

e. All leading 1's in a matrix in row echelon form must occur in different columns.

f. If every column of a matrix in row echelon form has a leading 1, then all entries that are not leading 1's are zero.

g. If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.

h. If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

i. If a linear system has more unknowns than equations, then it must have infinitely many solutions.

Working with Technology

T1. Find the reduced row echelon form of the augmented matrix for the linear system

$$\begin{aligned} 6x_1 + x_2 \phantom{{}+3x_3} + 4x_4 &= -3 \\ -9x_1 + 2x_2 + 3x_3 - 8x_4 &= 1 \\ 7x_1 \phantom{{}+2x_2} - 4x_3 + 5x_4 &= 2 \end{aligned}$$

Use your result to determine whether the system is consistent and, if so, find its solution.

T2. Find values of the constants A, B, C, and D that make the following equation an identity (i.e., true for all values of x).

$$\frac{3x^3 + 4x^2 - 6x}{(x^2 + 2x + 2)(x^2 - 1)} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{C}{x - 1} + \frac{D}{x + 1}$$

[*Hint:* Obtain a common denominator on the right, and then equate corresponding coefficients of the various powers of x in the two numerators. Students of calculus will recognize this as a problem in partial fractions.]

1.3 Matrices and Matrix Operations

Rectangular arrays of real numbers arise in contexts other than as augmented matrices for linear systems. In this section we will begin to study matrices as objects in their own right by defining operations of addition, subtraction, and multiplication on them.

Matrix Notation and Terminology

In Section 1.2 we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate systems of linear equations. However, rectangular arrays of numbers occur in other contexts as well. For example, the following rectangular array with three rows and seven columns might describe the number of hours that a student spent studying three subjects during a certain week:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Math	2	3	2	4	1	4	2
History	0	3	1	4	3	2	2
Language	4	1	3	1	0	0	2

If we suppress the headings, then we are left with the following rectangular array of numbers with three rows and seven columns, called a "matrix":

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 4 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 & 2 \\ 4 & 1 & 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

More generally, we make the following definition.

Definition 1

A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** of the matrix.

EXAMPLE 1 | Examples of Matrices

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

Matrix brackets are often omitted from 1×1 matrices, making it impossible to tell, for example, whether the symbol 4 denotes the number "four" or the matrix [4]. This rarely causes problems because it is usually possible to tell which is meant from the context.

The ***size*** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. For example, the first matrix in Example 1 has three rows and two columns, so its size is 3 by 2 (written 3×2). In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices in Example 1 have sizes 1×4, 3×3, 2×1, and 1×1, respectively.

A matrix with only one row, such as the second in Example 1, is called a ***row vector*** (or a ***row matrix***), and a matrix with only one column, such as the fourth in that example, is called a ***column vector*** (or a ***column matrix***). The fifth matrix in that example is both a row vector and a column vector.

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities; thus we might write

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 4 & 2 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

When discussing matrices, it is common to refer to numerical quantities as ***scalars***. Unless stated otherwise, *scalars will be real numbers*; complex scalars will be considered later in the text.

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij}. Thus a general 3×4 matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

When a compact notation is desired, matrix (1) can be written as

$$A = [a_{ij}]_{m\times n} \quad \text{or} \quad A = [a_{ij}]$$

the first notation being used when it is important in the discussion to know the size, and the second when the size need not be emphasized. Usually, we will match the letter denoting a matrix with the letter denoting its entries; thus, for a matrix B we would generally use b_{ij} for the entry in row i and column j, and for a matrix C we would use the notation c_{ij}.

The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix (1) above, we have

$$(A)_{ij} = a_{ij}$$

and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2$, $(A)_{12} = -3$, $(A)_{21} = 7$, and $(A)_{22} = 0$.

Row and column vectors are of special importance, and it is common practice to denote them by boldface lowercase letters rather than capital letters. For such matrices, double subscripting of the entries is unnecessary. Thus a general $1 \times n$ row vector $\mathbf{a}$ and a general $m \times 1$ column vector $\mathbf{b}$ would be written as

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A matrix A with n rows and n columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (2) are said to be on the ***main diagonal*** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

Operations on Matrices

So far, we have used matrices to abbreviate the work in solving systems of linear equations. For other applications, however, it is desirable to develop an "arithmetic of matrices" in which matrices can be added, subtracted, and multiplied in a useful way. The remainder of this section will be devoted to developing this arithmetic.

Definition 2

Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

EXAMPLE 2 | Equality of Matrices

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

If $x = 5$, then $A = B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are the same. There is no value of x for which $A = C$ since A and C have different sizes.

Definition 3

If A and B are matrices of the same size, then the ***sum*** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference*** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

In matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad (A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

The equality of two matrices

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

of the same size can be expressed either by writing

$$(A)_{ij} = (B)_{ij}$$

or by writing

$$a_{ij} = b_{ij}$$

EXAMPLE 3 | Addition and Subtraction

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined.

Definition 4

If A is any matrix and c is any scalar, then the ***product*** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a ***scalar multiple*** of A.

In matrix notation, if $A = [a_{ij}]$, then

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

EXAMPLE 4 | Scalar Multiples

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \tfrac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$.

Thus far we have defined multiplication of a matrix by a scalar but not the multiplication of two matrices. Since matrices are added by adding corresponding entries and subtracted by subtracting corresponding entries, it would seem natural to define multiplication of matrices by multiplying corresponding entries. However, it turns out that such a definition would not be very useful. Experience has led mathematicians to the following definition, the motivation for which will be given later in this chapter.

Definition 5

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the ***product*** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add the resulting products.

EXAMPLE 5 | Multiplying Matrices

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 26 & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows:

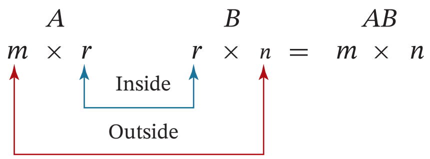
$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & 13 \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$\begin{aligned}(1\cdot 4)+(2\cdot 0)+(4\cdot 2)&=12\\(1\cdot 1)-(2\cdot 1)+(4\cdot 7)&=27\\(1\cdot 4)+(2\cdot 3)+(4\cdot 5)&=30\\(2\cdot 4)+(6\cdot 0)+(0\cdot 2)&=8\\(2\cdot 1)-(6\cdot 1)+(0\cdot 7)&=-4\\(2\cdot 3)+(6\cdot 1)+(0\cdot 2)&=12\end{aligned} \qquad AB=\begin{bmatrix}12 & 27 & 30 & 13\\ 8 & -4 & 26 & 12\end{bmatrix}$$

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB. If this condition is not satisfied, the product is undefined. A convenient way to determine whether a product of two matrices is defined is to write down the size of the first factor and, to the right of it, write down the size of the second factor. If, as in (3), the inside numbers are the same, then the product is defined. The outside numbers then give the size of the product.



(3)

EXAMPLE 6 | Determining Whether a Product Is Defined

Suppose that A, B, and C are matrices with the following sizes:

A	B	C
3×4	4×7	7×3

Then, AB is defined and is a 3×7 matrix; BC is defined and is a 4×3 matrix; and CA is defined and is a 7×4 matrix. The products AC, CB, and BA are all undefined.

In general, if $A=[a_{ij}]$ is an $m\times r$ matrix and $B=[b_{ij}]$ is an $r\times n$ matrix, then, as illustrated by the shading in the following display,

$$AB=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1r}\\ a_{21} & a_{22} & \cdots & a_{2r}\\ \vdots & \vdots & & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{ir}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mr}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n}\\ \vdots & \vdots & & \vdots & & \vdots\\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn}\end{bmatrix} \tag{4}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$\boxed{(AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\cdots+a_{ir}b_{rj}} \tag{5}$$

Formula (5) is called the ***row-column rule*** for matrix multiplication.

Partitioned Matrices

A matrix can be subdivided or ***partitioned*** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For example, the following are